NON-GAUSSIAN DISTRIBUTION IN HIGHER MOMENTS OF MATRIX ELEMENTS

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ABSTRACT. In this preprint, we study the distribution in higher moments of matrix elements. It turns out that the higher moments of matrix elements does not follow the gaussian distribution.

1. INTRODUCTION

Let $\mu_j(\psi) = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \psi(z)|\phi_j(z)|^2 d\mu$ where ψ, ϕ are even Hecke-Maass forms, ψ fixed and ϕ_j varying. Denote by $\lambda_j = \frac{1}{4} + t_j^2$ the Laplace eigenvalue of ϕ_j. When averaging over a suitable window of λ_j's, the local Weyl law says that the mean of the μ_j's is zero and the variance is the classical variance twisted by the central value of Hecke-Maass L-function of $\psi(z)$ [12]. Now normalize $\mu_j(\psi)$ by setting

$$F_j := \lambda_j^{1/4}\mu_j(\psi)$$

which have constant variance $\sigma(\psi)^2 > 0$ (assuming $L(\frac{1}{2}, \psi) \neq 0$). For generic system, Eckhart et al [3] predict a central limit theorem for F_j, i.e., the distribution of F_j is Gaussian. A stronger form is to say that the higher moments are Gaussian, that is

$$\frac{1}{N(T,H)} \sum_{T<\lambda_j<T+H} |F_j|^{2r} \sim \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x^{2r} e^{-x^2/2\sigma^2} dx$$

where $N(T,H) = \#\{T < \lambda_j < T+H\}$ and $H = T^a$ (the exact choice of window does not matter here and in fact we can choose a Gaussian window with $H = T^{1-\delta}$ as in [12]).

For the modular group, we expect that this Gaussian moments predict does not hold, in fact it is shown that the higher moments blow up, precisely we have

Theorem 1. *For even integer $r \geq 2$,*

$$\frac{1}{N(T,H)} \sum_{T<\lambda_j<T+H} |F_j|^{2r} \gg (\log T)^{r(r-1)/2}.$$

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To prove this, recall Watson's formula

$$|F_j|^2 \sim c(\psi)\frac{L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)}{L(1, \mathrm{Sym}^2\phi_j)^2}.$$

Therefore understanding moments of the normalized matrix coefficients is equivalent to studying moments of central values of the L-function $L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)$, decorated by the factor $1/L(1, \mathrm{Sym}^2\phi_j)^2$, as we vary over the family of all Hecke-Maass forms. Thus the problem falls into the realm of showing lower bounds for moments of L-functions, for which we can apply the technique in [8], [9].

Theorem 2. *For even integer $r \geq 2$,*

$$\frac{1}{N(T,H)} \sum_{T<\lambda_j<T+H} L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^r \gg (\log T)^{r(r-1)/2}.$$

To prove this, let

$$L^T(\frac{1}{2}, \psi \times \mathrm{sym}^2(\phi_j)) \quad = \quad \sum_{(m_1 m_2^2) \leq T^\epsilon} \lambda_\psi(m_1) a_{\Phi_j}(m_1, m_2)(m_1 m_2^2)^{-1/2}$$

we consider

$$S_1 := \sum_{T<\lambda_j<T+H} (L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j))^r,$$

$$S_2 := \sum_{T<\lambda_j<T+H} L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)(L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j))^{r-1}.$$

Keeping in mind that r is even so

$$|L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)|^r = (L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j))^r.$$

By Hölder inequality, we have

$$\sum_{T<\lambda_j<T+H} L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j) L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^{r-1}$$

$$\leq \quad \left(\sum_{T<\lambda_j<T+H} L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^r\right)^{\frac{1}{r}} \left(\sum_{T<\lambda_j<T+H} (L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^{r-1})^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}$$

Hence

$$\sum_{T<\lambda_j<T+H} L(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^r \geq \frac{S_2^r}{S_1^{r-1}}.$$

We can derive Theorem 2 by finding the following asymptotic orders of magnitude of S_1 and S_2:

Lemma 1. *For even integer* $r \geq 2$,

$$S_1 \asymp H(\log T)^{r(r-1)/2}.$$

Lemma 2. *For even integer* $r \geq 2$,

$$S_2 \asymp H(\log T)^{r(r-1)/2}.$$

2. Proof of Lemma 1

To evaluate

$$S_1 = \sum_{T<\lambda_j<T+H} L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^r,$$

we expand out $L^T(\frac{1}{2}, \psi \times \mathrm{Sym}^2\phi_j)^r$ and group terms using the Hecke relations. To do this conveniently, we denote by $\mathcal{H}$ the ring generated over the integers by symbols $x(n)$ subject to the Hecke relations

$$x(1) = 1, \qquad \text{and} \qquad x(m)x(n) = \sum_{d|(m,n)} x\left(\frac{mn}{d^2}\right).$$

Then $\mathcal{H}$ is a polynomial ring on $x(p)$ where p runs over all primes. By the Hecke relation we can write

$$x(n_1)\cdots x(n_r) = \sum_{t|n_1\cdots n_r} b_t(n_1, \cdots, n_r)x(t).$$

Obviously, the integer coefficients $b_t(n_1, \cdots, n_r)$ is symmetric in $n_1, \cdots, n_r$, and it is always non-negative. Also, we have

$$b_t(n_1, \cdots, n_r) \leq \tau(n_1)\cdots\tau(n_r) \ll (n_1\cdots n_r)^\epsilon$$

for any ϵ.

The coefficient of $x(1)$, $b_1(n_1, \cdots, n_r)$ is important in our proof. Here are some basic properties:

Multiplicative property: if $(\prod_{j=1}^r m_j, \prod_{j=1}^r n_j) = 1$, then

$$b_1(m_1n_1, \cdots, m_rn_r) = b_1(m_1, \cdots, m_r)b_1(n_1, \cdots, n_r).$$

So it suffices to understand $b_1(p^{a_1}, \cdots p^{a_r})$ for prime p. We have

$$0 \leq b_1(p^{a_1}, \cdots p^{a_r}) \leq (1+a_1)\cdots(1+a_r)$$

and it is 0 if $a_1 + \cdots + a_r$ is odd. If we let

$$B_r(n) = \sum_{n_1\cdots n_r=n} b_1(n_1, \cdots, n_r),$$

$B_r(n)$ is a multiplicative function and $B_r(n) = 0$ unless n is a square. Also, $B_r(p^a)$ is independent of p and grows at most polynomially in a. To be noted that

$$B_r(p^2) = \binom{r}{2} = \frac{r(r-1)}{2},$$

which follows by the facts $b_1(p^2, 1, \cdots, 1) = 0$ and $b_1(p, p, 1, \cdots, 1) = 0$.

Since

$$\begin{aligned} L^T(\frac{1}{2}, \psi \times \mathrm{sym}^2(\phi_j)) &= \sum_{(m_1 m_2^2) \leq T^\epsilon} \lambda_\psi(m_1) a_{\Phi_j}(m_1, m_2)(m_1 m_2^2)^{-1/2} \\ &= \sum_d \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1, s_2, t_1, t_2} \lambda_\psi(d s_1^2 t_1)(s_1^2 t_1 s_2^4 t_2^2)^{-1/2} \\ &\quad \lambda_j(t_1^2)\lambda_j(t_2^2) \end{aligned}$$

Here, we let $m_i = d s_i^2 t_i$. In view of Kuznetsov formula we will apply later, we analyze the above sum as follows: $m_1 m_2^2 = d^3 s_1^2 s_2^4 t_1 t_2^2$ essentially is $d^3 s_1^2$ since only small t_1, t_2 contribute. s_2^4 can be omitted, also.

For

$$\sum_{d \geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1} \frac{\lambda_\psi(d s_1^2)}{s_1}$$

use multiplication property, Euler product and shift contour, we can control the outer sums in the following $L^T(\frac{1}{2}, \psi \times \mathrm{sym}^2(\phi_j))^r$.

$$\begin{aligned} L^T(\frac{1}{2}, \psi \times \mathrm{sym}^2(\phi_j))^r &= \left(\sum_{(m_1 m_2^2) \leq T^\epsilon} \lambda_\psi(m_1) a_{\Phi_j}(m_1, m_2)(m_1 m_2^2)^{-1/2} \right)^r \\ &= \sum_{d_1, \cdots, d_r} \frac{\prod_1^r \mu(d_i)}{\prod_1^r d_i^{\frac{3}{2}}} \sum_{\substack{s_{1,1}, \cdots, s_{r,2} \\ t_{1,1}, \cdots, t_{r,2}}} \prod_1^r \lambda_\psi(d_i s_{i,1}^2 t_{i,1})(s_{i,1}^2 t_{i,1} s_{i,2}^4 t_{i,2}^2)^{-1/2} \\ &\quad \prod_1^r \lambda_j(t_{i,1}^2)\lambda_j(t_{i,2}^2) \\ &= \sum_{d_1, \cdots, d_r} \frac{\prod_1^r \mu(d_i)}{\prod_1^r d_i^{\frac{3}{2}}} \sum_{\substack{s_{1,1}, \cdots, s_{r,2} \\ t_{1,1}, \cdots, t_{r,2}}} \prod_1^r \lambda_\psi(d_i s_{i,1}^2 t_{i,1})(s_{i,1}^2 t_{i,1} s_{i,2}^4 t_{i,2}^2)^{-1/2} \\ &\quad \sum_{t | t_{1,1}^2 \cdots t_{r,2}^2} b_t(t_{1,1}^2, \cdots, t_{r,2}^2)\lambda_j(t) \end{aligned}$$

Hence we need evaluate $\sum_{T<\lambda_j<T+H} \lambda_j(t)$, to do this we apply the following proposition established by Kuznetsov formula:

Proposition 1. *Let s and t be positive integers with $st \leq T^2/100$, we have*

$$\sum_{T<\lambda_j<T+T^{1-\epsilon}} \lambda_j(s)\lambda_j(t) = T^{1-\epsilon}\delta(s,t) + O(e^{-T}),$$

where $\delta(s,t) = 1$ if $s = t$ and is 0 otherwise.

Since $t_{i,j} \leq T^\epsilon$, from the above proposition, we have the essential terms in S_1:

$$\sum_{n\leq T^\epsilon} B_{2r}(n) \leq \sum_{t_{1,1},\cdots,t_{2,r}\leq T^\epsilon} b_1(t_{1,1},\cdots,t_{2,r}) \leq \sum_{n\leq T^{2r\epsilon}} B_{2r}(n)$$

Where $B_r(n)$ is a multiplication function with $B_r(p) = 0$ and $B_r(p^2) = \dfrac{r(r-1)}{2}$, $B_r(p^n)$ grows only polynomially in n, so the generating function $\sum_{n=1}^{\infty} B_r(n)n^{-s}$ can be compared with $\zeta(2s)^{r(r-1)/2}$, the quotient being a Dirichlet series absolutely convergent in $\mathrm{Re}(s) > \dfrac{1}{4}$. Then by a standard argument [10], (applying Perron formula and shift contour)

$$\sum_{n\leq T^\epsilon} B_{2r}(n) \sim C_r T^\epsilon (\log T)^{r(r-1)/2}$$

Hence, we can deduce Lemma 1.

3. Proof of Lemma 2

To prove Lemma 2, we apply the similar argument as the proof in Lemma 1 and an approximate functional equation for $L(\frac{1}{2}, \psi\times\mathrm{Sym}^2\phi_j)$:

$$L(1/2, \psi\otimes\mathrm{sym}^2(\phi_j)) = 2\sum_{m_1,m_2\geq 1} \lambda_\psi(m_1)a_{\Phi_j}(m_1,m_2)(m_1m_2^2)^{-1/2}V(\frac{m_1m_2^2}{t_j^2})$$

where

$$V(y) = \frac{1}{2\pi i}\int_{(2)} y^{-s}\frac{\gamma(1/2+s,\psi\otimes\mathrm{sym}^2(\phi_j))}{\gamma(1/2,\psi\otimes\mathrm{sym}^2(\phi_j))}\frac{ds}{s},$$

$$\begin{aligned}\gamma(s,\psi\otimes\mathrm{sym}^2(\phi_j)) \quad = \quad & \pi^{-3s}\Gamma\left(\frac{s+it_\psi}{2}+it_j\right)\Gamma\left(\frac{s+it_\psi}{2}-it_j\right)\Gamma\left(\frac{s+it_\psi}{2}\right) \\ & \Gamma\left(\frac{s-it_\psi}{2}+it_j\right)\Gamma\left(\frac{s-it_\psi}{2}-it_j\right)\Gamma\left(\frac{s-it_\psi}{2}\right).\end{aligned}$$

We have the following proposition:

Proposition 2.

$$L(1/2,\psi\otimes \mathrm{sym}^2(\phi_j)) = 2\sum_{m_1m_2^2\le T}\lambda_\psi(m_1)a_{\Phi_j}(m_1,m_2)(m_1m_2^2)^{-1/2}V_T(\frac{m_1m_2^2}{t_j^2}) + O(e^{-T})$$

For the weight $V_T(\xi)$, it satisfies $|V_T(\xi)|\ll T\pi^{-T}/\xi$ for $\xi>T$, $V_T(\xi)=1+O(e^{-T})$ for small $\xi<T/100$ and $V_T(\xi)\ll 1$ for $T/100\le\xi\le T$.

Combined with the method in Lemma 1 and the observation that

$$b_1(n_1,\cdots,n_{2r-1},t)=b_t(n_1,\cdots,n_{2r-1})$$

if $t|n_1\cdots n_{2r-1}$, otherwise $b_1(n_1,\cdots,n_{2r-1},t)=0$. We may see Lemma 2 follows.

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