Boundary values of Thurston's pullback map

Complex analysis, dynamics, and geometry seminar (UMich)

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Nov 23, 2015

Thurston maps

Let $F : S^2 \longrightarrow S^2$ be a degree d branched cover. Denote the critical set by C_F. The *postcritical set* is given by

$$P_F = \bigcup_{i>0} F^{\circ i}(C_F).$$

If $|P_F|$ is finite, F is said to be *postcritically finite*.

Definition

A *Thurston map* is a postcritically finite orientation-preserving branched cover $F : S^2 \longrightarrow S^2$ with $d \geq 2$.

Combinatorial equivalence

Let F and G be Thurston maps with postcritical sets P_F and P_G respectively.

Definition

F is *combinatorially equivalent* to G if the following commutes:

$$\begin{tikzcd} (S^2, P_F) \arrow[r, "h_1"] \arrow[d, "F"'] & (S^2, P_G) \arrow[d, "G"] \\ (S^2, P_F) \arrow[r, "h_0"] & (S^2, P_G) \end{tikzcd}$$

where h_0 and h_1 are orientation preserving homeomorphisms so that h_0 is homotopic to h_1 rel P_F.

Background for Thurston's theorem

- A simple closed curve γ in $S^2 \setminus P_F$ is *essential* if each component of $S^2 \setminus \gamma$ intersects P_F in at least two points.
- A *multicurve* is a collection $\Gamma = \{\gamma_1, ...\gamma_k\}$ of disjoint essential curves where the elements of the collection are pairwise non-homotopic.
- Let $\mathscr{C}_F$ denote the set of homotopy classes of essential simple closed curves in $S^2 \setminus P_F$. Denote by $\mathbb{R}[\mathscr{C}_F]$ the free $\mathbb{R}$-module over $\mathscr{C}_F$.

Associated to each Thurston map F is a linear map $\lambda_F : \mathbb{R}[\mathscr{C}_F] \longrightarrow \mathbb{R}[\mathscr{C}_F]$ defined by:

$$\lambda_F(\gamma) = \sum_{\gamma'} \sum_{\gamma' \simeq \delta \subset F^{-1}(\gamma)} \frac{1}{\deg(F : \delta \to \gamma)} \cdot \gamma'$$

where γ and γ' are essential curves and the outer sum is over all γ' homotopic to preimages of γ.

Definition

An *obstruction* is a nonempty multicurve Γ so that $\mathbb{R}[\Gamma]$ is invariant under λ_F, and the spectral radius of λ_F is greater than or equal to 1.

Thurston's theorem

Theorem (W. Thurston)

Let F be a Thurston map with hyperbolic orbifold. Then F is equivalent to a rational function if and only if there are no obstructions. If this rational function exists, it is unique up to Möbius conjugation.

Ingredients in the proof: F is equivalent to a rational map $\iff$ the pullback map on Teichmüller space has a fixed point. Uniqueness follows from the fact that the pullback map is contracting in the Teichmüller metric.

Teichmüller space

Definition

The *Teichmüller space* associated to a Thurston map F is defined as follows:

$$\mathscr{T}_F := \{\text{orientation pres. homeos } \phi : (S^2, P_F) \to \widehat{\mathbb{C}}\}/\sim$$

where $\phi_1 \sim \phi_2$ if there exists a Möbius transformation M so that ϕ_2 is isotopic to $M \circ \phi_1$ relative to P_F.

Definition

The *moduli space* associated to F is given by

$$\mathscr{M}_F := \{\iota : P_F \hookrightarrow \widehat{\mathbb{C}}\}/\sim$$

where $\iota_1 \sim \iota_2$ if there is a Möbius transform M so that $M \circ \iota_1 = \iota_2$.

Thurston's pullback map

Let $[\tau] \in \mathscr{T}_F$. We have the following commutative diagram:

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{\tilde{\tau}} & (\widehat{\mathbb{C}}, \tilde{\tau}(P_F)) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_\tau} \\ (S^2, P_F) & \xrightarrow{\tau} & (\widehat{\mathbb{C}}, \tau(P_F)) \end{array}$$

where F_τ is rational.

Definition

The pullback map $\sigma_F : \mathscr{T}_F \to \mathscr{T}_F$ is given by $\sigma_F([\tau]) = [\tilde{\tau}]$.

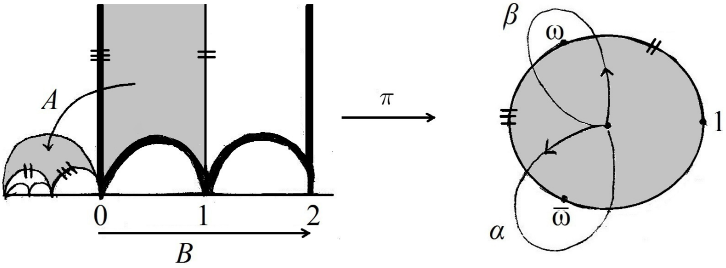
The Weil-Petersson metric

Some facts about the WP metric on Teichmüller space:

- The pure mapping class group $\mathrm{PMCG}(S^2, P_F)$ acts on $\mathscr{T}_F$ by isometry.
- $\mathscr{T}_F$ under the the WP metric is not complete
- The action of $\mathrm{PMCG}(S^2, P_F)$ extends to the completion

Special case: $|P_F| = 4$

$\mathrm{PMCG}(S^2, P_F) = \mathrm{P}\Gamma(2) = \langle A, B\rangle$ where $A = \left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$.



The WP boundary is $\mathbb{Q} \cup \{1/0\}$. Neighborhood basis elements are given by the horoball topology. Boundary points correspond to essential curves.

The pullback on curves (when $|P_f| = 4$)

Definition

The *pullback function on curves* $\mu_F : \mathscr{C}_F \cup \{\odot\} \to \mathscr{C}_F \cup \{\odot\}$ is defined as follows:

- $\mu_F(\gamma) = \tilde{\gamma}$ if $F^{-1}(\gamma)$ has some essential component $\tilde{\gamma}$
- $\mu_F(\gamma) = \odot$ otherwise

Theorem (Selinger)

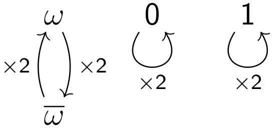
σ_F can be extended continuously to the Weil-Petersson boundary of Teichmüller space. Furthermore $\sigma_F(S_\gamma) \subset S_{\mu_F(\gamma)}$.

Extended analysis of an example

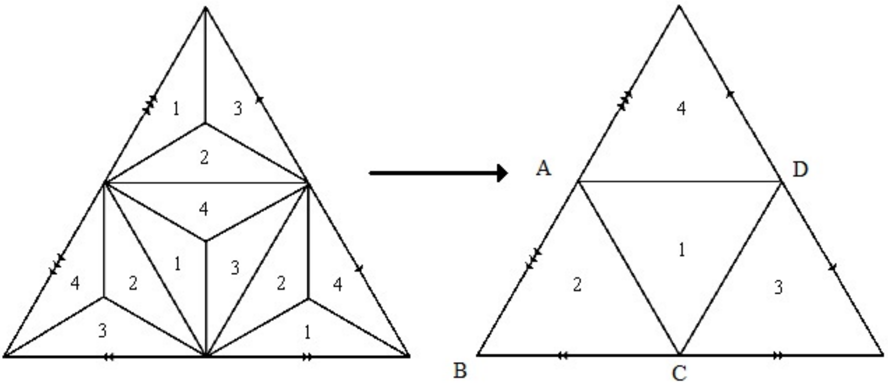
A Thurston map first studied by Buff, Epstein, et al.:

$$f(z) = \frac{3z^2}{2z^3 + 1}$$

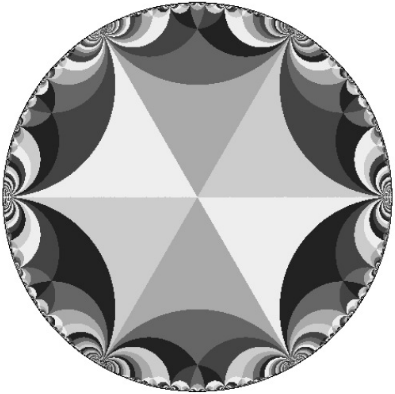
$P_f = \{0, 1, \omega, \overline{\omega}\}$ where $\omega = e^{2\pi i/3}$.
$C_f = \{0, 1, \omega, \overline{\omega}\}$



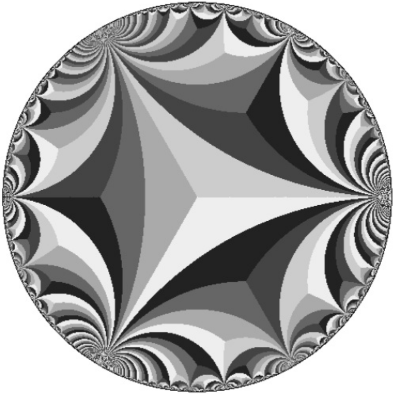
Subvision model for f



Pullback map on Teichmüller space



$\xrightarrow{\sigma_f}$



Correspondence on moduli space

For any Thurston map F, there exists an intermediate complex manifold $\mathcal{W}_F$ so that the following diagram commutes (all maps are holomorphic, A, Y are covers), (Douady, Hubbard):

$$
\begin{tikzcd}
\mathscr{T}_F \arrow[rr, "\sigma_F"] \arrow[dr, "A"] \arrow[dd, "\pi"'] & & \mathscr{T}_F \arrow[dd, "\pi"] \\
& \mathscr{W}_F \arrow[dl, "Y"] \arrow[dr, "X"'] & \\
\mathscr{M}_F & & \mathscr{M}_F .
\end{tikzcd}
$$

Correspondence for example

$$X(z) = z^2 \qquad Y(z) = \frac{z(z^3+2)}{2z^3+1}$$

$$A(\tau) = \frac{x^2 - y}{2xy - 2} \text{ where } y = \pi(\tau), \quad x = \pi \circ \sigma_f(\tau)$$

$$\begin{tikzcd}
\mathscr{T}_f \arrow[rrr, "\sigma_f"] \arrow[ddd, "\pi"'] \arrow[drr, "A"] & & & \mathscr{T}_f \arrow[ddd, "\pi"] \\
& & \widehat{\mathbb{C}} - \Theta' \arrow[ddl, "Y"'] \arrow[ddr, "X"] & \\
& & & \\
\widehat{\mathbb{C}} - \Theta & & & \widehat{\mathbb{C}} - \Theta
\end{tikzcd}$$

Θ = cube roots of unity
Θ' = sixth roots of unity

Wreath recursions on moduli space

Definition

The *wreath product* $G \wr S_d$ for some group G is given by $G^d \rtimes S_d$ where S^d acts on G^d by permutation of coordinates. Specifically multiplication is given by:

$$\langle g_1, \ldots, g_d \rangle \sigma \langle h_1, \ldots, h_d \rangle \tau = \langle g_1 h_{\sigma(1)}, \ldots, g_d h_{\sigma(d)} \rangle \sigma\tau$$

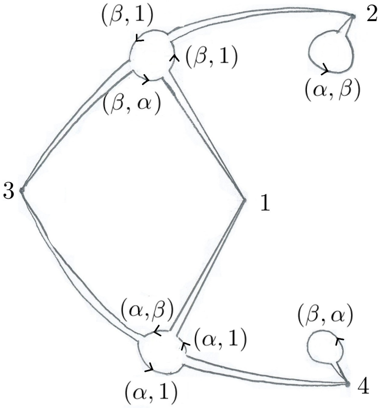
Definition

A *wreath recursion* is a homomorphism $\Phi : G \to G \wr S_d$.

Wreath recursion for the correspondence

$$\Phi(\beta) = \langle 1, 1, \alpha, \alpha \rangle (1 \quad 2 \quad 3)$$
$$\Phi(\alpha) = \langle \beta, \beta, 1, 1 \rangle (1 \quad 3 \quad 4)$$

Dual Moore automaton:



Virtual endomorphism

For $x \in X = \{1, ..., d\}$, denote the restriction to the xth coordinate of $\Phi(g)$ by $g|_x$. For $v \in X^*$ and $x \in X$, inductively define $g|_{xv} := (g|_x)|_v$.

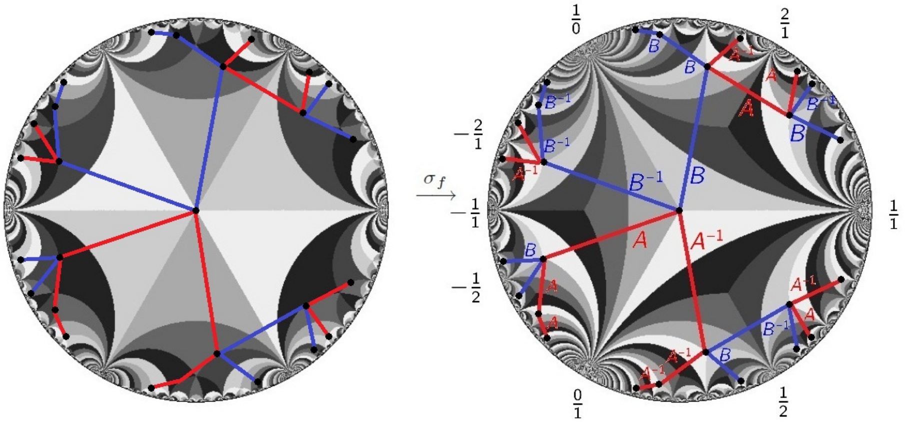
Definition

A *virtual endomorphism* $\phi : H \dashrightarrow G$ is a group homomorphism $\phi : H \to G$ where H is a finite index subgroup of G.

Note: $\Phi|_1 : H \to \pi_1(\mathscr{M}_F, 0)$ is a virtual endomorphism where:

$$H = \{[\gamma] \in \pi_1(\mathscr{M}_F) : \gamma \text{ lifts to a loop } \tilde{\gamma} \text{ based at 0 under } Y\}.$$

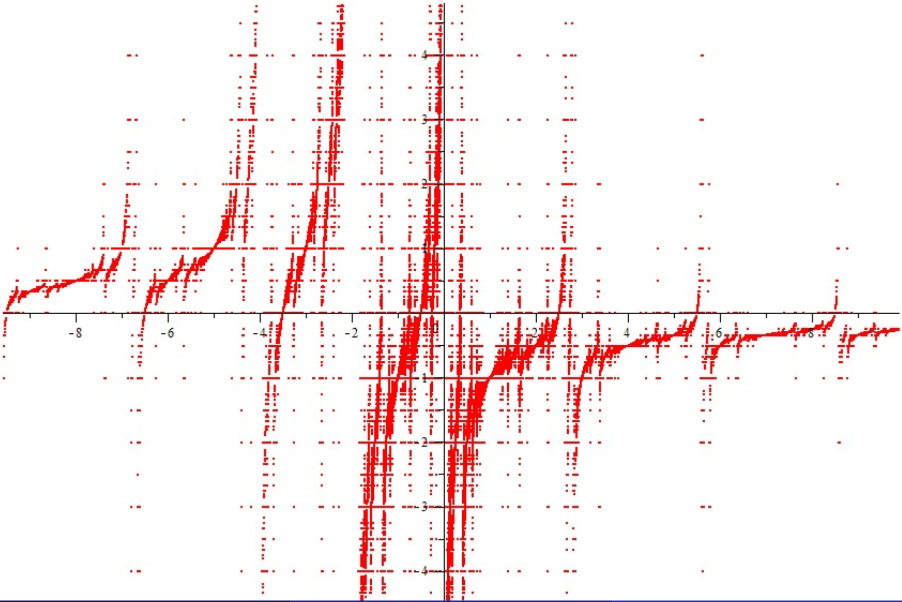
Algebraic model of Teichmüller space



Theorem (Lodge)

The algebraic model coincides with $\overline{\sigma_f}$ on $\partial_{WP}\mathscr{T}_f$.

Slope function for f



Results

- The graph is dense in $\mathbb{R}^2$.
- The fiber of μ_f over every point is infinite
- μ_f is surjective.

Theorem (Lodge, 2012)

Let $\frac{p}{q} \in \mathbb{Q} \cup \{1/0\}$ be a reduced fraction. Then under iteration of μ_f, $\frac{p}{q}$ lands either on the two-cycle $\frac{0}{1} \leftrightarrow \frac{1}{0}$ or on the fixed point $\frac{1}{1}$. More precisely, $\frac{p}{q}$ lands on $\frac{1}{1}$ if and only if p and q are odd.

$$\frac{203}{356} \longmapsto -\frac{50}{33} \longmapsto -\frac{13}{6} \longmapsto \frac{6}{1} \longmapsto -\frac{1}{2} \longmapsto \frac{0}{1}$$
$$\frac{203}{354} \longmapsto -\frac{28}{19} \longmapsto -\frac{7}{4} \longmapsto -\frac{4}{1} \longmapsto \frac{1}{0}.$$

Pilgrim's conjecture

Definition (Finite global attractor)

The pullback function on curves is said to have a *finite global attractor* if there is a finite set $\mathcal{A} \subset \mathscr{C}_f$ so that for any γ there exists n so that $\mu_F^{\circ n}(\gamma) \in \mathcal{A} \cup \{\odot\}$.

Conjecture

Suppose a postcritically finite rational map f has hyperbolic orbifold. Then the pullback on curves has a finite global attractor.

The converse is false.

Results

Theorem (Pilgrim 2012)

If the virtual endomorphism on moduli space is contracting, then the pullback function on multicurves has a finite global attractor.

Theorem (Pilgrim 2012)

Suppose f is a quadratic polynomial whose finite critical point is periodic. Then the pullback function on multicurves has a finite global attractor.

Theorem (Kelsey, Lodge 2015)

Suppose f is a rational quadratic map with hyperbolic orbifold and $|P_f| = 4$. Then the pullback function on curves has a finite global attractor.

NET maps

Definition (NET map)

A Thurston map is said to be *nearly Euclidean* if it has exactly four postcritical points, and all critical points are simple.

Examples: rabbit polynomial, airplane polynomial, basilica mate basilica, $f(z) = \frac{3z^2}{2z^3+1}$

NETmap output for rabbit

HALF-SPACE DATA

p/q	p'/q'	c	d	Center/ x-value	Shading	C-R	C+R
1/0	-1/1	1	2	-1.000000	In	-2.414214	0.414214
-4/1	-1/2	1	1	0.133333	In	-0.333333	0.600000
-3/1	0/1	1	2	0.333333	In	0.097631	0.569036
-2/1		0	1				
-1/1	0/1	1	2	1.000000	In	0.292893	1.707107
0/1	1/0	1	1	0.000000	Out	-1.000000	1.000000
1/1	-2/1	1	2	2.000000	Out	-0.121320	4.121320
2/1		0	1				
3/1	-2/1	1	2	-0.571429	In	-1.076505	-0.066352
4/1	-3/2	1	1	-1.428571	In	-3.000000	0.142857
-3/2	-1/1	1	2	0.647059	In	0.563870	0.730248
-1/2	1/1	1	2	5.000000	In	0.757359	9.242641
1/2	-3/1	1	2	1.000000	Out	-0.414214	2.414214
3/2	-1/1	1	2	-0.764706	In	-1.180651	-0.348761
-4/3	-1/2	1	1	0.666667	In	0.333333	1.000000
-2/3		0	1				
-1/3	2/1	1	2	-4.000000	Out	-8.949747	0.949747
1/3	-4/1	1	2	0.714286	Out	-0.598913	2.027484
2/3		0	1				
4/3	-3/2	1	1	-2.571429	In	-5.000000	-0.142857
-3/4	1/1	1	2	1.470588	In	0.888265	2.052911
-1/4	3/1	1	2	-1.571429	Out	-4.197825	1.054968
1/4	-5/1	1	2	0.565217	Out	-0.726021	1.856456
3/4	-3/1	1	2	-3.000000	In	-5.357023	-0.642977

SLOPE FUNCTION CYCLES

There are no fixed points p/q with |p| <= 4 and |q| <= 4.

NONTRIVIAL CYCLES
1/0 -> -1/1 -> 0/1 -> 1/0

The slope function orbit of every slope p/q with |p| <= 4
and |q| <= 4 ends in either one of these cycles or
a slope for which the slope function is undefined.

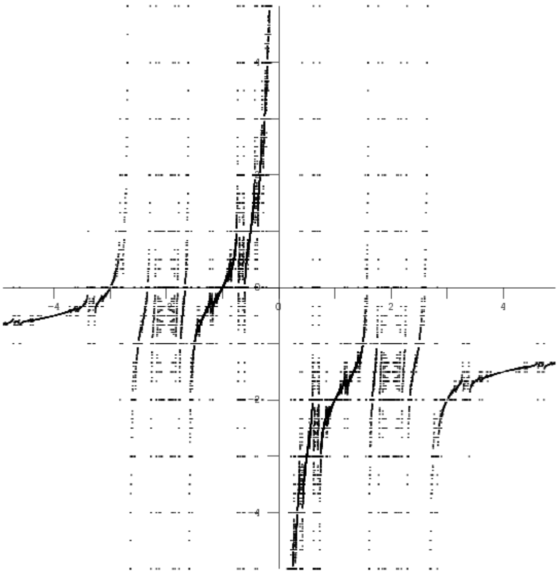
EXCLUDED INTERVALS FOR THE HALF-SPACE COMPUTATION
(-infinity, infinity)

The half-space computation determines rationality.
The extended half-space computation is not needed.

Every NET map in this pure modular group Hurwitz class is rational.

Press enter to close the terminal.

Slope function for rabbit



Application to polynomial mating

Definition

An equator for a Thurston map F is a Jordan curve γ in $S^2 \setminus P_F$ so that $F^{-1}(\gamma)$ is orientation-preserving isotopic to γ rel P_F and it maps by maximal degree.

Movie credit: Arnaud Chéritat

Theorem

A hyperbolic postcritically finite rational map arises as a mating $\iff$ *it has an equator*

NETmap reports that there is a hyperbolic rational NET map f of degree 506 that has at least 78 equators.

Application to twisting problems

Recall that $\pi_1(\widehat{\mathbb{C}} \setminus \Theta) = \langle \alpha, \beta \rangle$ can be identified with $\mathrm{PMCG}(\widehat{\mathbb{C}}, P_f) = \langle T_\alpha, T_\beta \rangle$ via the point-pushing isomorphism. For $g \in \pi_1(\widehat{\mathbb{C}} \setminus \Theta, 0)$, we define a right action of π_1 on f via $T_g \circ f = f \cdot g$

Problem (Twisting problem)

What is the combinatorial class of $f \cdot g$ where $g \in \pi_1(\widehat{\mathbb{C}} \setminus \Theta, 0)$?

Solution to the twisting problem

Following Bartholdi and Nekrashevych, extend $\psi : H \to \pi_1(\widehat{\mathbb{C}} \setminus \Theta)$ to a map

$$\overline{\psi} : \pi_1(\widehat{\mathbb{C}} \setminus \Theta) \longrightarrow \pi_1(\widehat{\mathbb{C}} \setminus \Theta)$$

defined as follows:

$$\overline{\psi}(g) = \begin{cases} \psi(g) & g \in H \\ \alpha\psi(g\alpha^{-1}) & g \in H\alpha \\ \alpha^{-1}\psi(g\alpha) & g \in H\alpha^{-1} \\ \beta\psi(g\beta^{-1}) & g \in H\beta. \end{cases}$$

Lemma

The Thurston map $f \cdot g$ is equivalent to $f \cdot \overline{\psi}(g)$.

Proof.

First suppose that $g \in H$. Then

$$T_g \circ f = f \circ T_{\psi(g)}.$$

Since $(f \circ T_{\psi(g)})^{T_{\psi(g)}} = T_{\psi(g)} \circ f$, one obtains

$$T_g \circ f \sim T_{\psi(g)} \circ f$$

Next suppose that $g \in H\alpha^{-1}$.

$$f \cdot g = \psi(g\alpha) \cdot f \cdot \alpha^{-1} \sim f \cdot \alpha^{-1}\psi(g\alpha) = f \cdot \overline{\psi}(g)$$

Theorem

Let $g \in \pi_1(\widehat{\mathbb{C}} \setminus \Theta)$. Then there is an N so that for all $n > N$, $\overline{\psi}^{\circ n}(g)$ is contained in the following set:

$$\mathfrak{M} = \{e, \beta, \alpha^{-1}, \alpha^2\beta^{-1}, \alpha^{-1}\beta\alpha^{-1}, \alpha\beta^{-1}, \beta^2\} \cup \{\alpha(\beta\alpha)^k : k \in \mathbb{Z}\}$$

Proof: Group theoretic analysis of MANY cases

Analysis of mystery maps

- Each member of the one parameter family is obstructed (since $\overline{\psi}$ fixes the twist)
- Each member of the finite list is unobstructed (obstructions must be Levy cycles, and GAP can shows they don't exist)
- Each member of the finite list must then be equivalent to the original f or a second rational map g
- To determine which, note that the pullback relation for f and g have different dynamical behavior

For instance, we conclude that $f \cdot \beta$ is combinatorially equivalent to g because:

$$\sigma_{f\cdot\beta}(\frac{1}{1}) = -\frac{1}{1}$$
$$\sigma_{f\cdot\beta}(-\frac{1}{1}) = \frac{1}{1}$$

and

$$\sigma_{f\cdot\beta}(\frac{1}{0}) = \frac{0}{1}$$
$$\sigma_{f\cdot\beta}(\frac{0}{1}) = \frac{1}{0}.$$

Thank you for your attention!