A combinatorial characterization of postcritically finite Newton maps

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Joint work with Y. Mikulich, D. Schleicher

Motivation

There is a bijection between the following:

- postcritically finite polynomials (up to affine conjugacy)
- minimal abstract extended Hubbard trees (up to equivalence)

Marked branched covers

$F : S^2 \longrightarrow S^2$ finite degree branched cover
C_F is the set of critical points

- The *postcritical set* of F is given by

$$P_F = \bigcup_{i>0} F^i(C_F).$$

- If $|P_F|$ is finite, F is said to be *postcritically finite*

Definition

A *marked branched cover* is a pair (F, X) where $F : S^2 \to S^2$ is a branched cover with $deg(F) \geq 2$, and X is a finite forward invariant set containing the postcritical set.

Marked branched covers

Definition

Let (F, X) and (G, Y) be marked branched covers. They are *Thurston equivalent* if there are orientation preserving homeomorphisms

$$h_0, h_1 : (S^2, X) \longrightarrow (S^2, Y)$$

with h_0 homotopic to h_1 rel X, so that the following commutes:

$$\begin{tikzcd} (S^2, X) \arrow[r, "h_1"] \arrow[d, "F"'] & (S^2, Y) \arrow[d, "G"] \\ (S^2, X) \arrow[r, "h_0"] & (S^2, Y) \end{tikzcd}$$

Newton Maps

Definition

A rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 3$ is a *Newton map* if there exists a polynomial p such that for every $z \in \mathbb{C}$,

$$f(z) = z - p(z)/p'(z)$$

Theorem (Y. Mikulich, D. Schleicher,–)

There is a bijection between the set of postcritically finite Newton maps up to affine conjugacy and the set of abstract extended Newton graphs up to Thurston equivalence.

Newton Facts

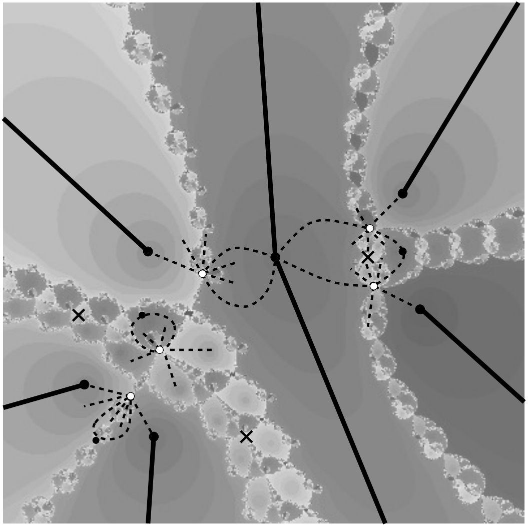
p is a degree d polynomial with simple roots
$N_p(z) = z - \frac{p(z)}{p'(z)}$

- N_p is a rational function of degree d
- $N_p'(\infty) = d/(d-1)$ and is a repelling fixed point
- The roots of p correspond to finite superattracting fixed points of N_p (since $N_p' = \frac{p \cdot p''}{p' \cdot p'}$)
- A degree d rational map has $d+1$ fixed points and $2d-2$ critical points counting multiplicity

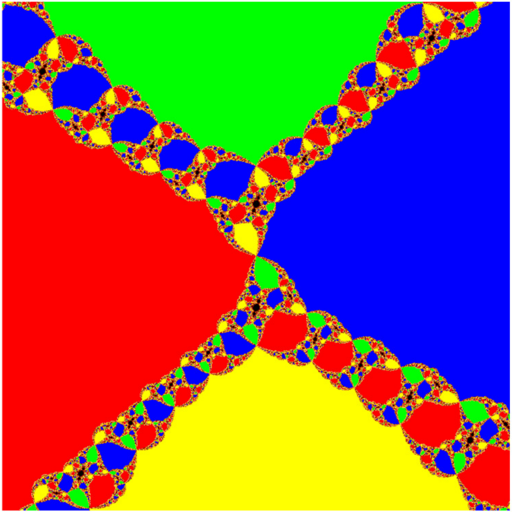
Thus the only fixed points of N_p are at ∞ and roots of p.

(Hubbard, Schleicher, Sutherland): The immediate basin of a fixed critical point of multiplicity m has exactly m accesses to infinity.

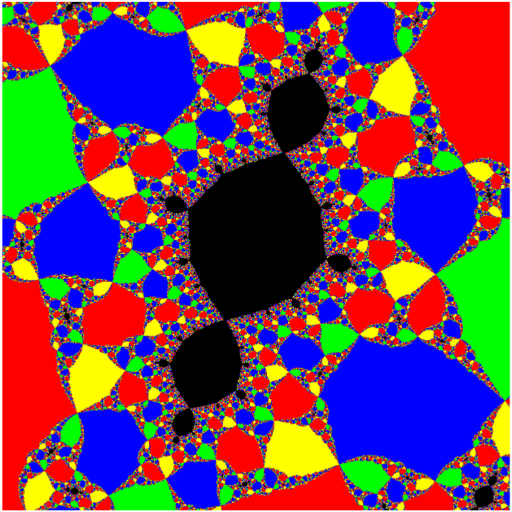
The Julia set for a degree 6 Newton map



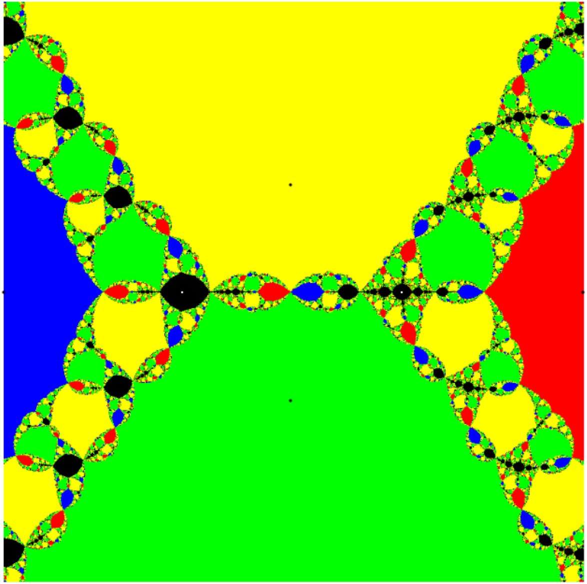
Degree 4 PCF Newton map (K.Mamayusupov)



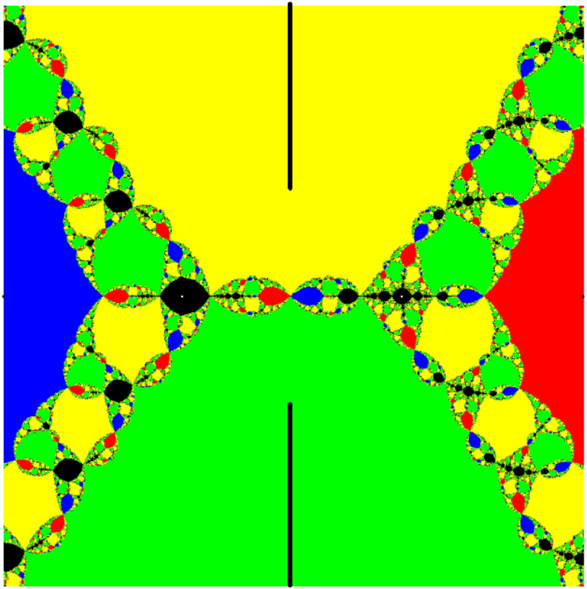
Detail of previous slide



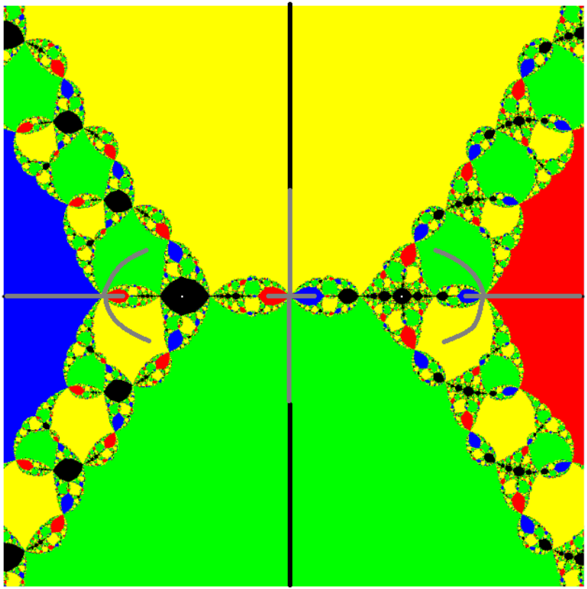
Degree 4 Newton map (K.M, L.Bartholdi)



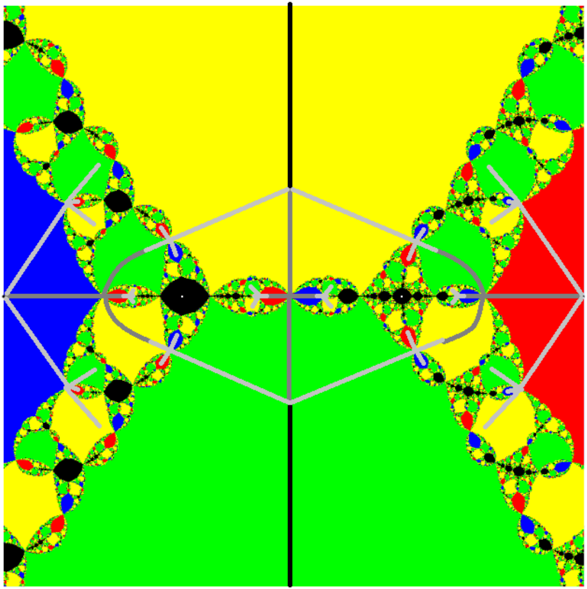
The Julia set for a Newton map



The Julia set for a Newton map



The Julia set for a Newton map



Definition (Newton graph of level n)

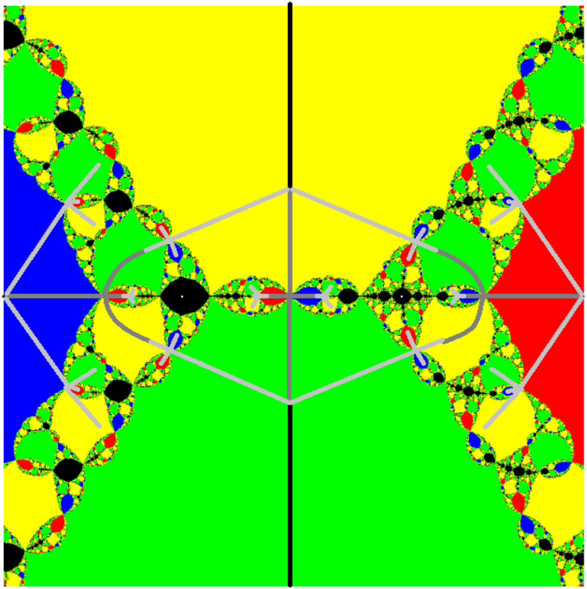
For $n \geq 0$,denote by Δ_n the component of $N_P^{-n}(\Delta)$ that contains Δ. The *Newton graph of level n* is the pair (Δ_n, N_p).

Theorem (Mikulich,Ruckert,Schleicher)

There is some level N of the Newton graph so that

- *All non-free critical points are contained in Δ_N.*
- $\overline{\Delta_N \setminus \Delta}$ *is connected*
- *The restriction of some iterate of N_p to all complementary components of Δ_N containing free critical points is renormalizable.*

The Julia set for a Newton map



Definition (Abstract Channel Diagram)

An *abstract channel diagram* of degree $d \geq 3$ is a graph $\Delta \subset S^2$ with vertices $v_1, \ldots, v_d, v_\infty$ and edges $e_1, \ldots, e_l$ that satisfies the following properties:

(1) $l \leq 2d - 2$

(2) each edge joins v_∞ to a v_i

(3) each v_i is connected to v_∞ by at least one edge

(4) if e_i and e_j both join v_∞ to v_k, then each connected component of $S^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ

Definition (Abstract Newton Graph)

Let $\Gamma \subset S^2$ be a connected finite graph and $f : \Gamma \to \Gamma$ a graph map. The pair (Γ, f) is called an *abstract Newton graph* if it satisfies the following conditions:

(1) There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that f fixes each vertex and each edge of Δ.

(2) The graph map f can be extended to a branched covering $f : S^2 \to S^2$ such that the following conditions (3) – (6) are satisfied.

Definition (Abstract Newton Graph cont'd)

(3) If $v_1, \ldots, v_{d_\Gamma}, v_\infty$ are the vertices of Δ, then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq \infty$. Moreover, there are exactly $\deg_{v_i}(\overline{f}) - 1 \geq 1$ edges in Δ that connect v_i to v_∞ for $i \neq \infty$.

(4) $\sum_{x \in \Gamma'} \left(\deg_x(\overline{f}) - 1\right) \leq 2d_\Gamma - 2$.

(5) The graph $\overline{\Gamma \setminus \Delta}$ is connected.

(6) Γ equals the component of $\overline{f}^{-N_\Gamma}(\Delta)$ that contains Δ.

Definition (Abstract Extended Newton Graph)

$\Sigma \subset S^2$ a finite connected graph, $f : \Sigma \to \Sigma$ a graph map, and Σ' the set of vertices. A pair (Σ, f) is called an *abstract extended Newton graph* if:

(1) (Edge Types) Any two different edges in Σ may only intersect at vertices of Σ. Every edge must be one of the following three types:
- An edge in the abstract Newton graph Γ
- An edge in a periodic or preperiodic abstract Hubbard tree
- A periodic or preperiodic abstract Newton ray

(2) (Abstract Newton graph) There exists a positive integer N and an abstract Newton graph Γ at level N so that $\Gamma \subseteq \Sigma$. Furthermore N is minimal so that condition (6) holds.

(3) (Unique extendability) f can be extended to a branched covering $\overline{f} : S^2 \to S^2$, unique up to Thurston equivalence

(4) (Topological admissibility) The total number of critical points of $\overline{f}$ counted with multiplicity is $2d_\Gamma - 2$, where d_Γ is the degree of the abstract channel diagram $\Delta \subset \Gamma$.

Definition (Abstract Extended Newton Graph (cont'd))

(5) (Periodic Hubbard trees) There is a finite collection of distinct, disjoint, possibly degenerate, abstract extended Hubbard trees $H_i \subset \Sigma$, where $H_i \cap \Gamma = \emptyset$, and for each H_i there is a minimal positive integer $m_i \geq 2$ called the period of the tree such that $\overline{f}^{m_i}(H_i) = H_i$. For all k, the forward image of $f^k(H_i)$ is always some periodic Hubbard tree.

(6) (Preperiodic Hubbard trees) There is a finite collection of distinct, disjoint, possibly degenerate abstract extended Hubbard trees $H_i' \subset \Sigma$ where $H_i' \cap \Gamma = \emptyset$, and for each i there is some minimal positive integer $\ell_i \geq 1$ so that $\overline{f}^{\ell_i}(H_i')$ is a periodic Hubbard tree. This ℓ_i is called the preperiod of H_i' and for $1 \leq k < \ell_i$, the forward image $f^k(H_i')$ is always some preperiodic Hubbard tree.

(7) (Trees separated) Any two different abstract extended Hubbard trees lie in different complementary components of Γ.

Definition (Abstract Extended Newton Graph (cont'd))

(8) (Periodic Newton Rays) For every periodic abstract extended Hubbard tree H_i of period m_i and every fixed point ω_i of H_i there exists a periodic abstract Newton ray $\mathcal{R}_i$ that lands at ω_i (note that it is not required that $\mathcal{R}_i \subset \Sigma$). For every β-fixed point ω_i of H_i, the graph Σ contains exactly one periodic abstract Newton ray $\mathcal{R}_i$ of period m_i that lands at ω_i, namely the rightmost such Newton ray.

(9) (Preperiodic Newton Rays) For every preperiodic tree H_i' of preperiod ℓ_i where $\overline{f}^{\ell_i}(H_i') = H_j$, the following holds: for every $\omega_i' \in H_i'$ so that $\overline{f}^{\ell_i}(\omega_i')$ is a β-fixed point of H_j, there exists exactly one preperiodic abstract Newton ray in Σ that lands at ω_i', and whose image under $\overline{f}^{\ell_i}$ is a ray of period m_j landing on H_j.

Background for Thurston's Theorem

- A simple closed curve γ in $S^2 \setminus X$ is *essential* if each component of $S^2 \setminus \gamma$ intersects X in at least two points.
- A *multicurve* is a collection $\Gamma = \{\gamma_1, ...\gamma_k\}$ of disjoint essential curves where the elements of the collection are pairwise non-homotopic.
- Use $\mathscr{C}_F$ to denote the set of homotopy classes of essential simple closed curves in $S^2 \setminus X$.
- Denote by $\mathbb{R}[\mathscr{C}_F]$ the free $\mathbb{R}$-module over $\mathscr{C}_F$.

Background for Thurston's Theorem

- The *Thurston linear map* $\lambda_F : \mathbb{R}[\mathscr{C}_F] \longrightarrow \mathbb{R}[\mathscr{C}_F]$ is defined by

$$\lambda_F(\gamma) = \sum_{\gamma'} \sum_{\gamma' \simeq \delta \subset F^{-1}(\gamma)} \frac{1}{\deg(F : \delta \to \gamma)} \cdot \gamma'$$

  where γ and γ' are essential curves and the outer sum is over all γ' homotopic to preimages of γ.
- A *Thurston obstruction* is a nonempty multicurve Γ so that $\mathbb{R}[\Gamma]$ is invariant under λ_F, and the spectral radius of λ_F is greater than or equal to 1.

Theorem (Thurston)

Let (F, X) be a marked branched cover not equivalent to a Lattès map. Then (F, X) is Thurston equivalent to a rational function if and only if there are no obstructions. If this rational function exists, it is unique up to Möbius conjugation.

Definition (Intersection Number)

Let α and β each be an arc or a simple closed curve in (S^2, X). Their *intersection number* is

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha,\, \beta' \simeq \beta} \#\{(\alpha' \cap \beta') \setminus X\}\ .$$

The intersection number extends bilinearly to arc systems and multicurves.

We will now prove that every abstract extended Newton graph gives rise to a postcritically finite Newton map.

Proof: Let Π be an irreducible Thurston obstruction, i.e. for $\gamma, \gamma' \in \Pi$, there exists $n > 0$ so that some component of $\overline{f}^{-n}(\gamma)$ is homotopic to γ' rel the vertices of Σ. We will show this results in a contradiction.

- Case 1: $\Pi \cdot \Delta \neq 0$
- Case 2: $\Pi \cdot \Delta = 0$

A set of pairwise non-isotopic arcs in (S^2, X) is called an *arc system*. Two arc systems Λ, Λ' are *isotopic* if each curve in Λ is isotopic relative to X to a unique element of Λ' and vice versa.

Denote by $\tilde{\Lambda}(f^{\circ n})$ the union of those components of $f^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X, and define $\tilde{\Pi}(f^{\circ n})$ similarly.

Theorem (Kevin Pilgrim, Tan Lei)

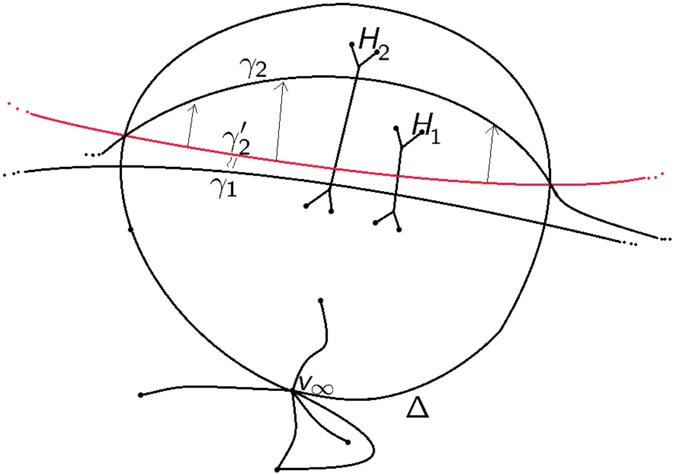
Let (f, X) be a marked branched covering, Π an irreducible Thurston obstruction and Λ an irreducible arc system. Suppose furthermore that $\#(\Pi \cap \Lambda) = \Pi \cdot \Lambda$. Then, exactly one of the following is true:

1. $\Pi \cdot \Lambda = 0$ *and* $\Pi \cdot f^{-n}(\Lambda) = 0$ *for all* $n \geq 1$.
2. $\Pi \cdot \Lambda \neq 0$ *and for* $n \geq 1$*, each component of* Π *is isotopic to a unique component of* $\tilde{\Pi}(f^{\circ n})$*. The mapping* $f^{\circ n} : \tilde{\Pi}(f^{\circ n}) \to \Pi$ *is a homeomorphism and* $\tilde{\Pi}(f^{\circ n}) \cap (f^{-n}(\Lambda) \setminus \tilde{\Lambda}(f^{\circ n})) = \emptyset$.

This theorem implies that $\Pi \cdot (\Gamma \setminus \Delta) = \emptyset$, and that Π may not intersect the Newton rays.

Proof of Case 1

Π is an irreducible Thurston obstruction, $\gamma_1 \in \Pi$ so that $\gamma_1 \cdot \Delta \neq 0$



Proof of Case 2

There is a complementary component U of the Newton graph containing some component of the obstruction Π as well as a Hubbard tree of some period m.

Then $(f|_U)^m$ is obstructed. Contradiction.

Thank you for your attention!